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Smooth global solutions of the two dimensional Burgers equation

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Abstract

It is shown in the present paper that the two dimensional Burgers equation describing a quasi-plane wave in a viscous heat conducting fluid admits smooth global solutions, provided initial data are smooth and small. Solutions decay at infinite time in a prescribed single space direction like those of the one dimensional linear heat equation.

1. Introduction and main results

In this paper we discuss global existence and asymptotic decay estimates of solutions to the initial value problem of the two dimensional Burgers equation

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0, \tag{1.1}$$

$$u(0, x, y) = u_0(x, y), \tag{1.2}$$

for a scalar unknown function $u = u(t, x, y)$ of time $t \geq 0$ and position $(x, y) \in \mathbf{R}^2$. This equation was first derived by Kuznetsov [8] in the form of the three dimensional Burgers equation

$$(u_t + uu_x - u_{xx})_x + \Delta u = 0, \tag{1.3}$$

for a scalar unknown function $u = u(t, x, y, z)$ of time $t \geq 0$ and position $(x, y, z) \in \mathbf{R}^3$, where $\Delta = \partial_y^2 + \partial_z^2$ and y and z are slowly varying transverse variables. Solutions to (1.3) describe a quasi-plane wave in the dynamics of a viscous heat conducting fluid, here a quasi-plane wave means a wave which propagates in almost one direction. Eq. (1.3) is reduced to (1.1) if, for example, the transverse direction appears only in one direction, $\Delta = \partial_y^2$, or if solutions to (1.3) exhibit a circular motion in the (y, z)-plane. In fact,

$$\Delta u = r^{-1}(ru_r)_r + r^{-2}u_{\theta\theta} = Cu_{\theta\theta},$$

where $y = r\cos\theta$, $z = r\sin\theta$, $C = r^{-2}$ and r is a positive constant. Eqs. (1.1) and (1.3) are basic ingredients in model equations describing a multidimensional quasi-plane wave. Other equations describing a multidimensional quasi-plane wave are the Zabolotskaya-Khokhlov (ZK) equation [13]

$$(u_t + uu_x)_x + u_{yy} = 0, \tag{1.4}$$

and the Kadomtsev-Petviashvili (KP) equation [7]

$$(u_t + uu_x \pm u_{xxx})_x + u_{yy} = 0. \tag{1.5}$$

These equations are also systematically obtained by the geometrical optics approximation [2], [3]. There have naturally been considerable physical and mathematical interest in the four equations (1.1), (1.3), (1.4) and (1.5). For all these equations, exact solutions have been obtained by several methods, such as the hodograph transformation [6], the similarity analysis [1], or the Painlevé analysis [10], [11]: On the other hand, for (1.5), a local existence theorem and a global existence theorem for small initial data are known [9], [12]: If $u_0 \in H^s(\mathbf{R}^2)$, $s \geq 3$, then a solution exists locally in time, and if $u_0 \in H^s(\mathbf{R}^2) \cap W^{s,1}(\mathbf{R}^2)$, $s \geq 10$, and u_0 is small, then a global solution exists.

In this paper we consider the two dimensional Burgers equation (1.1) and prove the existence of global-in-time solutions for initial data which are not necessarily of explicit form but are small in a Sobolev space. We can prove a similar global existence result also for the Burgers equation in higher space dimensions ((1.3) for example) but we omit it because its proof is essentialy the same as that in the two dimensional case.

To state our main results, we use the standard notations listed below.

Notations: Let N be a positive integer. For $p \in [1, \infty]$, $L^p(\mathbf{R}^N)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p(\mathbf{R}^N)}$. For integers $s \geq 0$, we denote by $W^{s,p}(\mathbf{R}^N)$ the space of functions $f = f(x)$ such that all the derivatives of f up to order s belong to $L^p(\mathbf{R}^N)$, with the norm

$$\|f\|_{W^{s,p}(\mathbf{R}^N)} = \left(\sum_{|\alpha| \leq s} \|\partial_x^\alpha f\|_{L^p(\mathbf{R}^N)} \right)^{1/p},$$

where $\alpha = (\alpha_1, \cdots, \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}$. When $p = 2$, we use the notation $H^s(\mathbf{R}^N)$ instead of $W^{s,p}(\mathbf{R}^N)$. We denote by $\mathcal{B}^k([0,\infty); W^{s,p}(\mathbf{R}^N))$ the space of functions $f = f(t)$ on $[0,\infty)$ such that $\partial_t^j f$, $0 \le j \le k$, are bounded and continuous from $[0,\infty)$ to $W^{s,p}(\mathbf{R}^N)$.

Let $\Omega = \mathbf{R}^2_{x,y}$, $\mathbf{R}_x \times \mathbf{T}_y$, $\mathbf{T}_x \times \mathbf{R}_y$ or $\mathbf{T}^2_{x,y}$, where $\mathbf{T} = \mathbf{R}/(2\pi\mathbf{Z})$ is an one dimensional torus, and let $\hat{\Omega} = \mathbf{R}^2_{\xi,\eta}$, $\mathbf{R}_\xi \times \mathbf{Z}_\eta$, $\mathbf{Z}_\xi \times \mathbf{R}_\eta$ or $\mathbf{Z}^2_{\xi,\eta}$. We denote the Fourier transform of $f(x,y) \in L^2(\Omega)$ by $\mathcal{F}_{x,y}[f](\xi,\eta)$, and the inverse Fourier transform of $g(\xi,\eta) \in L^2(\hat{\Omega})$ by $\mathcal{F}^{-1}_{\xi,\eta}[g](x,y)$. They are given as follows: When $\Omega = \mathbf{R}^2_{x,y}$ and $\hat{\Omega} = \mathbf{R}^2_{\xi,\eta}$,

$$\mathcal{F}_{x,y}[f](\xi,\eta) = \lim_{M\to\infty} \iint_\Omega \chi_{[-M,M]^2}(x,y) f(x,y) e^{-i(x\xi+y\eta)} dx dy \quad \text{in } L^2(\mathbf{R}^2_{\xi,\eta}),$$
$$\mathcal{F}^{-1}_{\xi,\eta}[g](x,y) = \frac{1}{(2\pi)^2} \lim_{M\to\infty} \iint_{\hat{\Omega}} \chi_{[-M,M]^2}(\xi,\eta) g(\xi,\eta) e^{i(x\xi+y\eta)} d\xi d\eta \quad \text{in } L^2(\mathbf{R}^2_{x,y}),$$

where and in what follows χ_A denotes the defining function of a set A. When $\Omega = \mathbf{T}_x \times \mathbf{R}_y$ and $\hat{\Omega} = \mathbf{Z}_\xi \times \mathbf{R}_\eta$,

$$\mathcal{F}_{x,y}[f](\xi,\eta) = \lim_{M\to\infty} \iint_\Omega \chi_{[-M,M]}(y) f(x,y) e^{-i(x\xi+y\eta)} dx dy \quad \text{in } L^2(\mathbf{R}_\eta),$$
$$\mathcal{F}^{-1}_{\xi,\eta}[g](x,y) = \frac{1}{(2\pi)^2} \lim_{M\to\infty} \int_{-M}^{M} \sum_{\xi\in\mathbf{Z}} g(\xi,\eta) e^{i(x\xi+y\eta)} d\eta \quad \text{in } L^2(\mathbf{R}_y).$$

We omit the formulas in the other two cases.

We consider (1.1) and (1.2) in $\Omega = \mathbf{R}^2$, $\mathbf{R}_x \times \mathbf{T}_y$, $\mathbf{T}_x \times \mathbf{R}_y$ and $\mathbf{T}^2$, where $\mathbf{T}$ corresponds to the periodic boundary condition. Let us transform (1.1) and (1.2) to an integral equation. To this end, we introduce an operator $U(t)$ as follows: For a function $f = f(x,y)$,

$$(U(t)f)(x,y) = \mathcal{F}^{-1}_{\xi,\eta}[q(t,\xi,\eta)\mathcal{F}_{x,y}[f](\xi,\eta)](x,y), \tag{1.6}$$

where

$$q(t,\xi,\eta) = \begin{cases} e^{-(\xi^2+i\frac{\eta^2}{\xi})t}, & \text{for } \xi \neq 0 \\ 1, & \text{for } \xi = 0. \end{cases} \tag{1.7}$$

By using $U(t)$, (1.1) and (1.2) are formally transformed to

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t \partial_x U(t-\tau) u(\tau)^2 d\tau. \tag{1.8}$$

In fact, when $\Omega = \mathbf{R}^2$ for example, applying the Fourier transform with respect to x and y to (1.1) and (1.2), we have

$$(\mathcal{F}_{x,y}[u])_t = -(\xi^2 + i\frac{\eta^2}{\xi})\mathcal{F}_{x,y}[u] - \frac{1}{2} i\xi \mathcal{F}_{x,y}[u^2],$$
$$\mathcal{F}_{x,y}[u](0,\xi,\eta) = \mathcal{F}_{x,y}[u_0](\xi,\eta).$$

Hence,

$$\begin{aligned}\mathcal{F}_{x,y}[u](t,\xi,\eta) &= e^{-(\xi^2+i\frac{\eta^2}{\xi})t}\mathcal{F}_{x,y}[u_0](\xi,\eta)\\ &\quad -\frac{1}{2}i\xi\int_0^t e^{-(\xi^2+i\frac{\eta^2}{\xi})(t-\tau)}\mathcal{F}_{x,y}[u^2](\tau,\xi,\eta)d\tau. \end{aligned} \tag{1.9}$$

Applying the inverse Fourier transform with respect to ξ and η to (1.9), we obtain (1.8). From now on we study the solvability of (1.8) and then consider the relationship between the solutions of (1.8) and of the original problem. Our main results are the following.

Theorem 1.1. *Let $\Omega = \mathbf{R}^2$, $\mathbf{R}_x \times \mathbf{T}_y$, $\mathbf{T}_x \times \mathbf{R}_y$ or $\mathbf{T}^2$.*

(i) (Uniqueness). Solutions of (1.8) are unique in $L^\infty([0,T);H^1(\Omega))$ for each $T > 0$.

(ii) (Local existence). Let $s \geq 1$ be an integer and let $u_0 \in H^s(\Omega)$. When $\Omega = \mathbf{T}_x \times \mathbf{R}_y$ or $\Omega = \mathbf{T}^2$, we also require

$$\int_{\mathbf{T}} u_0(x,y)dx = 0 \quad a.a.\ y. \tag{1.10}$$

Then, there is a constant $T_0 > 0$ depending only on $||u_0||_{H^s(\Omega)}$ such that there is a unique solution $u \in \mathcal{B}^0([0,T_0];H^s(\Omega))$ of (1.8).

Theorem 1.2. *(Global existence). Let $\Omega = \mathbf{R}^2$ and let $s \geq 1$ be an integer.*

(i) Suppose that $u_0 \in H^s(\Omega) \cap H^s(\mathbf{R}_y;L^1(\mathbf{R}_x))$. Then there is a constant $r_0 > 0$ such that if

$$M_0 \equiv ||u_0||_{H^s(\Omega)} + ||u_0||_{H^s(\mathbf{R}_y;L^1(\mathbf{R}_x))} < r_0, \tag{1.11}$$

then there is a unique solution $u \in \mathcal{B}^0([0,\infty);H^s(\Omega))$ of (1.8) satisfying

$$||\partial_x^k u(t)||_{L^2(\mathbf{R}_x;H^l(\mathbf{R}_y))} \leq CM_0(1+t)^{-\frac{k}{2}-\frac{1}{4}} \tag{1.12}$$

for integers k and l with $0 \leq k, l, k+l \leq s$, where C is a constant.

(ii) Suppose further that $xu_0 \in H^s(\mathbf{R}_y;L^1(\mathbf{R}_x))$ and

$$\int_{-\infty}^{\infty} u_0(x,y)dx = 0 \quad a.a.\ y. \tag{1.13}$$

Then there is a constant $r_1 > 0$ such that if

$$M_1 \equiv ||u_0||_{H^s(\Omega)} + ||xu_0||_{H^s(\mathbf{R}_y;L^1(\mathbf{R}_x))} < r_1, \tag{1.14}$$

then there is a unique solution $u \in \mathcal{B}^0([0,\infty);H^s(\Omega))$ of (1.8) satisfying

$$||\partial_x^k u(t)||_{L^2(\mathbf{R}_x;H^l(\mathbf{R}_y))} \leq CM_1(1+t)^{-\frac{k}{2}-\frac{3}{4}} \tag{1.15}$$

for integers k and l with $0 \leq k, l, k+l \leq s$, where C is a constant.

Remark 1.1. (i) A similar global existence result holds true also for $\Omega = \mathbf{R}_x \times \mathbf{T}_y$ (y-periodic case).

(ii) For the Fourier transform of the solution with respect to x, more detailed estimates hold. That is, for arbitrarily fixed $\alpha > 1$,

$$||(1+|\xi|^2 t)^\alpha (i\xi)^k \mathcal{F}_x[u](t)||_{L^2(\mathbf{R}_\xi;H^l(\mathbf{R}_y))} \le CM_\beta(1+t)^{-\frac{k+\beta}{2}-\frac{1}{4}} \tag{1.16}$$

and

$$\begin{aligned} &||(i\xi)^k \mathcal{F}_x[u](t,\xi)||_{H^l(\mathbf{R}_y)} \\ \le\ & |\xi|^k e^{-|\xi|^2 t} ||\mathcal{F}_x[u_0](\xi)||_{H^l(\mathbf{R}_y)} + CM_\beta^2 \rho(t,\xi)(1+|\xi|^2 t)^{-\alpha}(1+t)^{-(k+\beta)/2}, \end{aligned} \tag{1.17}$$

for integers k and l with $0 \le k, l, k+l \le s$, where $\beta = 0$ in Theorem 1.2 (i) and $\beta = 1$ in Theorem 1.2 (ii), and

$$\rho(t,\xi) = \begin{cases} (1+|\xi|^2 t)^{-1/2}, & \text{for } |\xi| \le 1 \\ (1+t)^{-1/2}(1+|\xi|^2)^{-1/2}, & \text{for } |\xi| \ge 1. \end{cases} \tag{1.18}$$

Note that

$$||\rho(t)||_{L^2(\mathbf{R}_\xi)} \le C(1+t)^{-1/4}. \tag{1.19}$$

The r_j, $j = 0, 1$, in (1.11) and (1.14) depend on α. Moreover, as an consequence of (1.16), we have

$$||\partial_x^{k+m} u(t)||_{L^2(\mathbf{R}_x;H^l(\mathbf{R}_y))} \le CM_\beta t^{-\frac{m}{2}}(1+t)^{-\frac{k+\beta}{2}-\frac{1}{4}}$$

for any integer m with $0 \le m \le 2\alpha$.

Theorem 1.3. *(Global existence of x-periodic solutions). Let $\Omega = \mathbf{T}_x \times \mathbf{R}_y$ and let $s \ge 1$ be an integer. Suppose that $u_0 \in H^s(\Omega)$ and (1.10). Then there is a constant $r_2 > 0$ such that if $||u_0||_{H^s(\Omega)} < r_2$, then there is a unique solution $u \in \mathcal{B}^0([0,\infty); H^s(\Omega))$ of (1.8) satisfying*

$$\int_{\mathbf{T}_x} u(t,x,y)dx = 0 \quad \text{for any } t \ge 0,\ y \in \mathbf{R} \tag{1.20}$$

and

$$||u(t)||_{H^s(\Omega)} \le C||u_0||_{H^s(\Omega)} e^{-\delta t}, \tag{1.21}$$

where $C > 0$ and $\delta \in (0,1)$ are constants.

Remark 1.2. (i) A similar global existence results holds true also for $\Omega = \mathbf{T}^2$.

(ii) For the Fourier transform of the solution with respect to x, more detailed estimates hold. That is, for arbitrarily fixed $\alpha > 1$,

$$||(1+|\xi|^2 t)^\alpha (i\xi)^k \mathcal{F}_x[u](t)||_{L^2(\mathbf{Z}_\xi;H^l(\mathbf{R}_y))} \le C||u_0||_{H^s(\Omega)} e^{-\delta t} \tag{1.22}$$

and for $\xi \in \mathbf{Z} \setminus \{0\}$,

$$\begin{aligned} \|(i\xi)^k \mathcal{F}_x[u](t,\xi)\|_{H^l(\mathbf{R}_y)} &\leq |\xi|^k e^{-|\xi|^2 t} \|\mathcal{F}_x[u_0](\xi)\|_{H^l(\mathbf{R}_y)} \\ &\quad + C(1+|\xi|^2)^{-1/2} e^{-\delta t}(1+|\xi|^2 t)^{-\alpha} \|u_0\|^2_{H^s(\Omega)} \end{aligned} \tag{1.23}$$

for integers k and l with $0 \leq k, l, k+l \leq s$. The r_2 in Theorem 1.3 depends on α.

Theorem 1.4. *(Differentiability in t). Let $\Omega = \mathbf{R}^2$ and let $s \geq 3$ be an integer. Suppose that u_0 satisfies the same assumptions in Theorem 1.2 (ii). Then the solution u in Theorem 1.2 (ii) of (1.8) also satisfies*

$$u \in \mathcal{B}^1([0,\infty); H^{s-2}(\mathbf{R}^2)) \tag{1.24}$$

and becomes a solution of the original pronlem (1.1) and (1.2). The time derivative of u is given by

$$\begin{aligned} u_t(t) &= \partial_x^2 U(t) u_0 - \lim_{M \to -\infty} \int_M^x (\partial_y^2 U(t) u_0)(x', y) dx' \\ &\quad - (u u_x)(t) - \frac{1}{2} \int_0^t (\partial_x^3 - \partial_y^2) U(t-\tau) u(\tau)^2 d\tau, \end{aligned} \tag{1.25}$$

where the limit is taken in $\mathbf{R}$. Similar results hold true also for $\Omega = \mathbf{R}_x \times \mathbf{T}_y$, $\mathbf{T}_x \times \mathbf{R}_y$ and $\mathbf{T}^2$.

Remark 1.3. Let a be any real constant state. If $u_0 - a$ is in a Sobolev space, then we can obtain a slight modification of Theorems 1.1-1.4. For example, the following counterpart of Theorem 1.1 (ii) holds true:

Let $s \geq 1$ be an integer. Suppose that $u_0 - a \in H^s(\Omega)$ when $\Omega = \mathbf{R}^2$ or $\mathbf{R}_x \times \mathbf{T}_y$, and that $u_0 - a \in H^s(\Omega)$ and $\int_{\mathbf{T}} (u_0(x,y) - a) dx = 0$ for any y when $\Omega = \mathbf{T}_x \times \mathbf{R}_y$ or $\Omega = \mathbf{T}^2$. Then there is a constant $T_0 > 0$ depending only on $\|u_0 - a\|_{H^s(\Omega)}$ such that there is a unique solution u of (1.8) with $u - a \in \mathcal{B}^0([0, T_0]; H^s(\Omega))$.

These theorems show that the solution of (1.8) behaves at infinite time in the x-direction like that of the linear heat equation $u_t - u_{xx} = 0$.

Finally it should be noted that some exact solutions found by Cates and Crighton [1] and by Webb and Zank [10] are not included in the class of solutions presented in this paper. The simplest example of such exact solutions is given by

$$u(t,x,y) = -2\frac{\theta_h}{\theta} + f'(t) y - f(t)^2,$$

where $h = x - f(t) y$ and $\theta = \theta(t,h)$ is a solution of

$$\theta_t - \theta_{hh} + (p(t) h + q(t))\theta = 0,$$

and $f(t)$, $p(t)$ and $q(t)$ are arbitrary functions depending only on t. For the details, see [10].

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